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Technical Note TN 35

Some remarks on the arithmetico- geometrical mean

by

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Introduction.

In this note we study some properties of the arithmetico-geometrical mean $M(a,b)$, which is defined as follows:
let a and b be two positive numbers,
let a_n and b_n be defined by the following recursive relations

$$(1) \quad a_{n+1} = (a_n+b_n)/2 \quad , \; b_{n+1} = (a_n b_n)^{1/2} \quad \text{with } a_0=a \text{ and } b_0=b,$$

then $$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n .$$

In an extensive study*) on the arithmetico-geometrical mean Gauss already derived the following two formulas which will be the starting points of our study.

$$(2) \quad \frac{\pi}{2\,M(a,b)} = \int_0^{\pi/2} (a^2\sin^2\varphi + b^2\cos^2\varphi)^{-1/2}\, d\varphi \quad \text{and}$$

$$(3) \quad \frac{\pi}{2\,M(a,b)} = 2\int_0^{\pi/2} (\alpha^2 - \beta^2\sin^2\varphi)^{-1/2}\, d\varphi$$

where $\alpha = a+b$, $\beta = a-b$.

In section 1 a series expansion for $\{M(1+k,1-k)\}^{-1}$ is derived valid for $|k| < 1$.
In section 2 a proof is given of the formula

$$(4) \quad \lim_{\varepsilon\to 0} M(1,\varepsilon)\log\frac{4}{\varepsilon} = \frac{\pi}{2}$$

In section 3 we present a relation between $M(a,b)$ and $M(\alpha,\beta)$ with $\alpha = a+b$ and $\beta = a-b$.

*) see C.F.Gauss: Anziehung eines elliptischen Ringes. Herausgegeben von H.Geppert. Akad. Verlags Gesellschaft, Leipzig 1927.

All results given here are well-known, but the treatment is possibly new.

1. <u>The series expansion of</u> $\left\{M(1+k, 1-k)\right\}^{-1}$

By aid of formula (2) we get for any ε, $0 < \varepsilon < 1$

$$\frac{1}{M(1,\varepsilon)} = \frac{1}{2\pi} \int_0^{2\pi} (\sin^2\varphi + \varepsilon^2\cos^2\varphi)^{-1/2}\, d\varphi =$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{i\varphi}\left\{(1+\varepsilon)e^{2i\varphi} - (1-\varepsilon)\right\}^{-1/2}\left\{-(1-\varepsilon)e^{2i\varphi} + (1+\varepsilon)\right\}^{-1/2} d\varphi\ .$$

Substituting $z = e^{i\varphi}$, we obtain the important formula

$$(5)\quad \frac{1}{M(1,\varepsilon)} = \frac{1}{\pi\sqrt{(1-\varepsilon^2)}} \oint_C \left\{(z^2 - \frac{1-\varepsilon}{1+\varepsilon})(z^2 - \frac{1+\varepsilon}{1-\varepsilon})\right\}^{-1/2} dz\ ,$$

where the integration must be performed in the positive sense along the contour C. C is the unit circle around the origin of the complex plane, which has cuts

from $-\infty$ to $-(\frac{1+\varepsilon}{1-\varepsilon})^{1/2}$,

from $-(\frac{1-\varepsilon}{1+\varepsilon})^{1/2}$ to $+(\frac{1-\varepsilon}{1+\varepsilon})^{1/2}$ and

from $+(\frac{1+\varepsilon}{1-\varepsilon})^{1/2}$ to $+\infty$ along the real axis, as shown in fig. 1 where

$$(6)\quad \xi = (\frac{1+\varepsilon}{1-\varepsilon})^{1/2} \quad \text{and} \quad \eta = (\frac{1-\varepsilon}{1+\varepsilon})^{1/2}$$

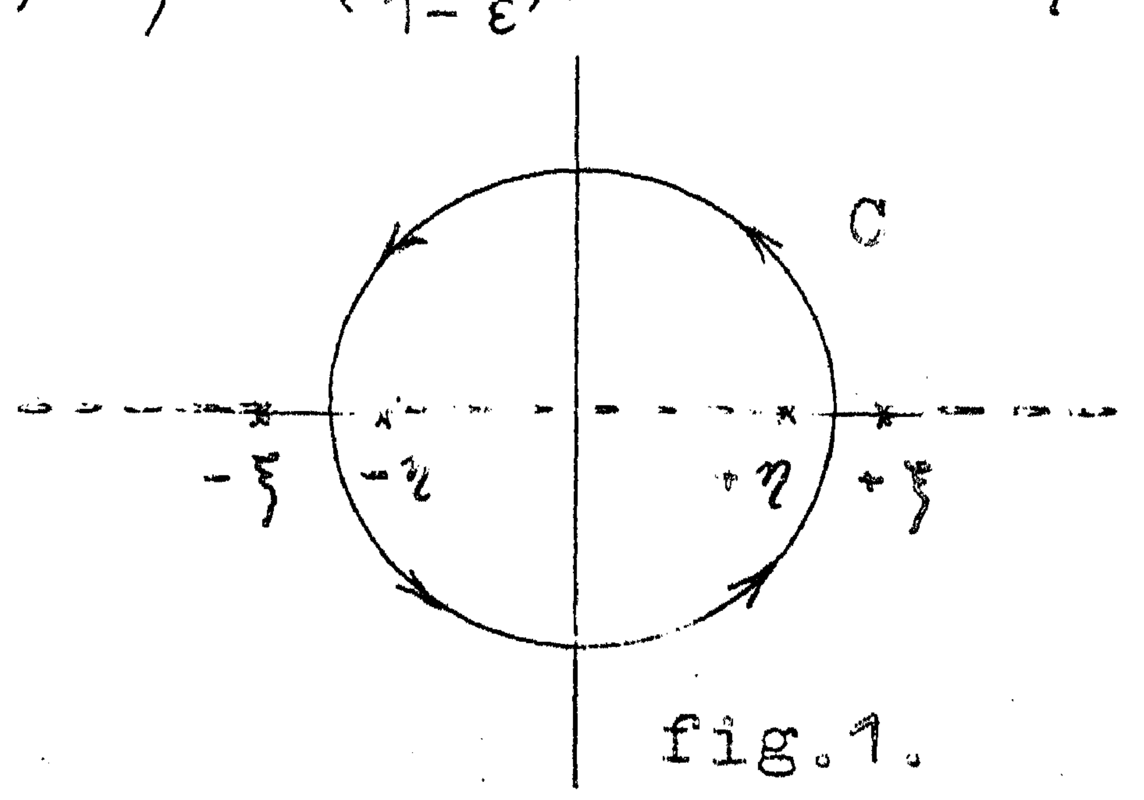


fig. 1.

We now deform the contour C into the contour K, as shown in fig. 2. K consists of the two contracting circles with centres $\pm\eta$ and the two lines parallel to the real axis.

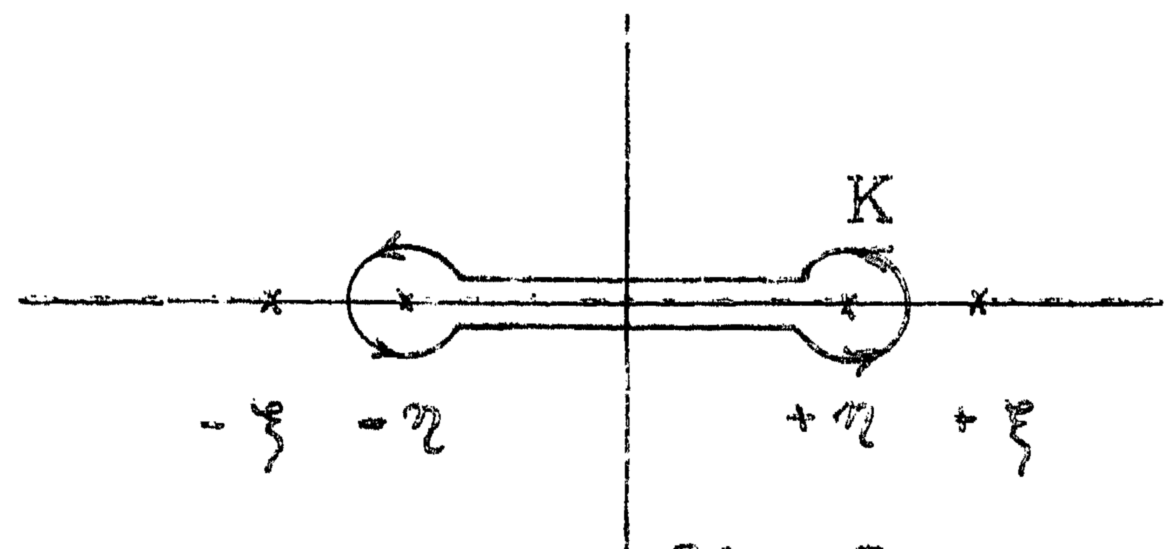


fig.2.

It is easily seen that for even values of n :

$$\frac{1}{2\pi i} \oint_K z^n (z^2-\eta^2)^{-1/2} dz = \eta^n \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-1}{n}$$

Therefore we obtain from (5)

$$\frac{1}{M(1,\varepsilon)} = \frac{2}{1+\varepsilon} \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu}^2 \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2\nu} ,$$

or, when we define k by $k = \frac{1-\varepsilon}{1+\varepsilon}$, we obtain

$$(7) \qquad \left\{M(1+k, 1-k)\right\}^{-1} = \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu}^2 k^{2\nu} .$$

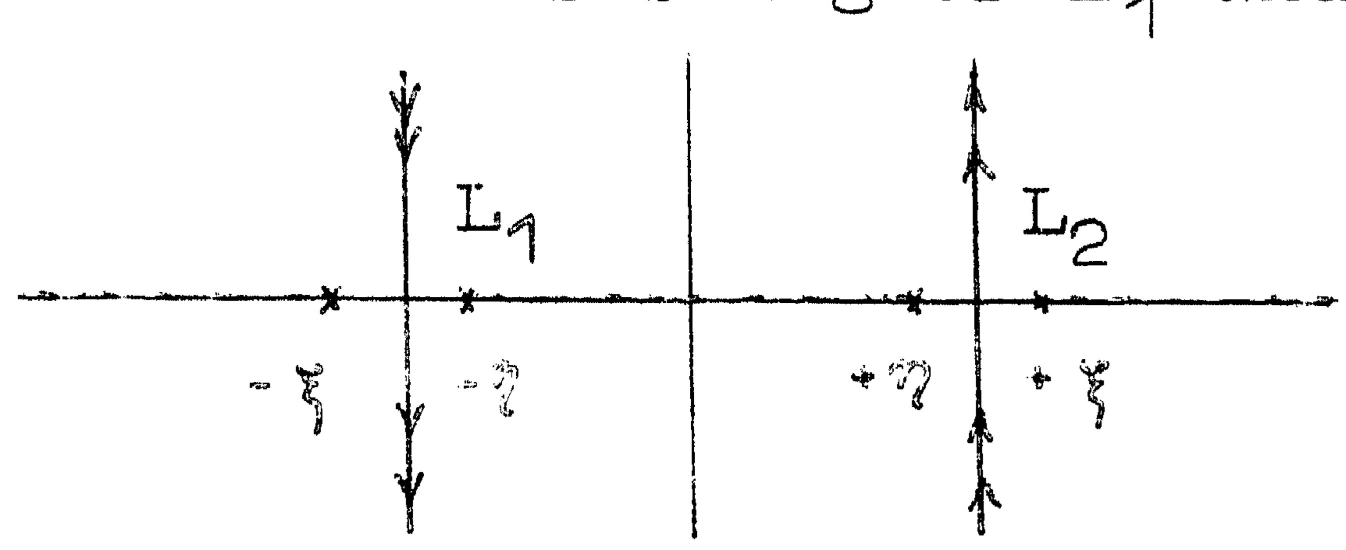
3. The limit expression.

In the following we use the well known formula

$$(8) \qquad \int_{C'} f(z)\,(z^2-x^2)^{-1/2}\, dz = \int_{-\infty}^{+\infty} f(ix\,\mathrm{sh}u)du ,$$

Where the path of integration C' is along the imaginary axis from $-i\infty$ to $+i\infty$, the complex plane has a cut along the real axis from $-\infty$ to $-x$ and from $+x$ to $+\infty$.
We assume the convergence of the integrals.

The contour C in formula (5) will now be deformed in a contour L consisting of L_1 and L_2 as shown in fig.3, where L_1 and L_2 intersect the real axis in the points $-(\xi+\eta)/2$ and $(\xi+\eta)/2$.



After this deformation of the path of integration we may write

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \left\{ \int_{-i\infty+\frac{\xi+\eta}{2}}^{+i\infty+\frac{\xi+\eta}{2}} - \int_{-i\infty-\frac{\xi+\eta}{2}}^{+i\infty-\frac{\xi+\eta}{2}} \right\} \left\{(z^2-\eta^2)(z^2-\xi^2)\right\}^{-1/2} dz =$$

$$= \int_{-i\infty}^{+i\infty} \left\{(z+\frac{\xi+\eta}{2})^2 - \eta^2\right\}^{-1/2} \left\{(z+\frac{\xi+\eta}{2})^2 - \xi^2\right\}^{-1/2} dz +$$

$$- \int_{-i\infty}^{+i\infty} \left\{(z-\frac{\xi+\eta}{2})^2 - \eta^2\right\}^{-1/2} \left\{(z-\frac{\xi+\eta}{2})^2 - \xi^2\right\}^{-1/2} dz =$$

$$= \int_{-i\infty}^{+i\infty} (z+\frac{\xi+3\eta}{2})^{-1/2} (z+\frac{\xi-\eta}{2})^{-1/2} (z+\frac{3\xi+\eta}{2})^{-1/2} (z-\frac{\xi-\eta}{2})^{-1/2} dz +$$

$$- \int_{-i\infty}^{+i\infty} (z-\frac{\xi+3\eta}{2})^{-1/2} (z-\frac{\xi-\eta}{2})^{-1/2} (z-\frac{3\xi+\eta}{2})^{-1/2} (z+\frac{\xi-\eta}{2})^{-1/2} dz$$

We put now: $\frac{\xi-\eta}{2} = \frac{1}{2}\left((\frac{1+\varepsilon}{1-\varepsilon})^{1/2} - (\frac{1-\varepsilon}{1+\varepsilon})^{1/2} \right) = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \gamma,$

$$\frac{\xi+3\eta}{2} = \frac{1}{2}\left((\frac{1+\varepsilon}{1-\varepsilon})^{1/2} + 3(\frac{1-\varepsilon}{1+\varepsilon})^{1/2} \right) = \frac{2-\varepsilon}{\sqrt{1-\varepsilon^2}} = \alpha,$$

$$\frac{3\xi+\eta}{2} = \frac{1}{2}\left(3(\frac{1+\varepsilon}{1-\varepsilon})^{1/2} + (\frac{1-\varepsilon}{1+\varepsilon})^{1/2} \right) = \frac{2+\varepsilon}{\sqrt{1-\varepsilon^2}} = \beta.$$

Hence we obtain:

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \int_{-i\infty}^{+i\infty} (z^2-\gamma^2)^{-1/2} (z+\alpha)^{-1/2} (z+\beta)^{-1/2} dz +$$

$$- \int_{-i\infty}^{+i\infty} (z^2-\gamma^2)(z-\alpha)^{-1/2} (z-\beta)^{-1/2} dz$$

$$= \int_{-\infty}^{+\infty} \left[\left\{ (i\gamma\ \mathrm{sh}u+\alpha)(i\gamma\ \mathrm{sh}u+\beta) \right\}^{-1/2} - \left\{ (i\gamma\ \mathrm{sh}u-\alpha)(i\gamma\ \mathrm{sh}u-\beta) \right\}^{-1/2} \right] du,$$

where we have used formula (8).

For small values of ε we have $\alpha = \beta + O(\varepsilon)$ and so we may write

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = \int_{-\infty}^{+\infty} \left\{ (i\gamma\ \mathrm{sh}u+\alpha)^{-1} - (i\gamma\ \mathrm{sh}u-\alpha)^{-1} \right\} du + r(\varepsilon)$$

Where $r(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Completing the reduction we obtain finally

$$\frac{\pi\sqrt{1-\varepsilon^2}}{M(1,\varepsilon)} = 2\alpha \int_{-\infty}^{+\infty} \frac{du}{\gamma^2 \mathrm{sh}^2 u + \alpha^2} + r(\varepsilon) =$$

$$= \frac{2}{\sqrt{\alpha^2-\gamma^2}} \ln \frac{\left\{\alpha + \sqrt{\alpha^2-\gamma^2}\right\}^2}{\gamma^2} + r(\varepsilon) =$$

$$= 2 \ln \frac{4}{\varepsilon} + r_1(\varepsilon)\ , \text{ where } r_1(\varepsilon) \to 0 \text{ for } \varepsilon \to 0$$

We have thus obtained the desired result

$$(4) \quad \lim_{\varepsilon \to 0} M(1,\varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2}\ .$$

3. <u>The relation between</u> M(a,b) <u>and</u> $M(\alpha,\beta)$.

We next study the relation between M(a,b) and $M(\alpha,\beta)$ where $\alpha = a+b$ and $\beta = a-b$.

From (3) we have

$$\frac{\pi}{2M(a,b)} = \frac{\pi}{2M(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2})} = 2\int_0^{\pi/2} (\alpha^2-\beta^2\sin^2\varphi)^{-1/2}\, d\varphi\ .$$

The second identity is also true for $\alpha_1 = (\alpha+\beta)/2$, and $\beta_1 = (\alpha\beta)^{1/2}$, thus

$$\frac{\pi}{2M\left(\frac{\alpha_1+\beta_1}{2}, \frac{\alpha_1-\beta_1}{2}\right)} = 2 \int_0^{\pi/2} (\alpha_1{}^2 - \beta_1{}^2 \sin^2\varphi)^{-1/2}\, d\varphi .$$

Evidently, $M(a,b) = M(a_1,b_1) = \dots = M(a_n,b_n)$ where a_n and b_n are defined according to (1), but

$$\left(\frac{\alpha_1+\beta_1}{2} + \frac{\alpha_1-\beta_1}{2}\right)/2 = \frac{1}{2}\left(\frac{\alpha+\beta}{2}\right) \quad \text{and}$$

$$\left(\frac{\alpha_1+\beta_1}{2} \cdot \frac{\alpha_1-\beta_1}{2}\right)^{1/2} = \frac{1}{2}\left(\frac{\alpha-\beta}{2}\right) \text{, so that}$$

$$M\left(\frac{\alpha_1+\beta_1}{2}, \frac{\alpha_1-\beta_1}{2}\right) = \frac{1}{2}\, M\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right).$$

Thus it follows immediately that

$$\frac{\pi}{2M(a,b)} = \frac{1}{2}\, \frac{\pi}{2M\left(\frac{\alpha_1+\beta_1}{2}, \frac{\alpha_1-\beta_1}{2}\right)} = 2 \cdot \frac{1}{2} \int_0^{\pi/2} (\alpha_1{}^2 - \beta_1{}^2 \sin^2\varphi)^{-1/2}\, d\varphi.$$

Repeating the above argument we find

$$(9) \quad \frac{\pi}{2M(a,b)} = 2\, \frac{1}{2^n} \int_0^{\pi/2} (\alpha_n{}^2 - \beta_n{}^2 \sin^2\varphi)^{-1/2}\, d\varphi ,$$

where α_n and β_n are defined by $\alpha_{n+1} = (\alpha_n+\beta_n)/2$,

$$\beta_{n+1} = (\alpha_n\beta_n)^{1/2} .$$

The integrand in (9) may be written as

$$(10) \quad (\beta_n{}^2 \cos^2\varphi + \gamma_n{}^2)^{-1/2}, \text{ with } \gamma_n{}^2 = \alpha_n{}^2 - \beta_n{}^2 .$$

Using Mc Laurins series we may write for this expression

$$\left\{\beta_n \cos\varphi + \frac{\gamma_n^{\,2}}{2\beta_n \cos\varphi} - \frac{\gamma_n^{\,4}}{8\beta_n^{\,3}\cos^3\varphi}\left(1 + \theta\,\frac{\gamma_n^{\,2}}{\beta_n^{\,2}\cos^2\varphi}\right)^{-3/2}\right\}^{-1}.$$

with $|\theta| < 1$

Let us now define

$$A = \beta_n \cos\varphi + \frac{\gamma_n^{\,2}}{2\beta_n \cos\varphi} \quad \text{and}$$

$$B = \frac{\gamma_n^{\,4}}{8\beta_n^{\,3}\cos^3\varphi}\left(1 + \theta\,\frac{\gamma_n^{\,2}}{\beta_n^{\,2}\cos^2\varphi}\right)^{-3/2}.$$

With this simplification formula (10) becomes

$$\frac{1}{A-B} = \frac{1}{A} + \frac{B}{A(A-B)}$$

and the integral (9) takes on the form

$$\frac{\pi}{2M(a,b)} = \frac{1}{2^{n-1}}\left[\int_0^{\pi/2-\gamma_n/\beta_n} \frac{1}{A}\,d\varphi + \int_0^{\pi/2-\gamma_n/\beta_n} \frac{B}{A(A-B)}\,d\varphi + \right.$$

$$\left. + \int_{\pi/2-\gamma_n/\beta_n}^{\pi/2} \left(\beta_n^{\,2}\cos^2\varphi + \gamma_n^{\,2}\right)^{-1/2} d\varphi\right].$$

The integrals appearing in this expression will be denoted consecutively by $I_{1,n}$, $I_{2,n}$ and $I_{3,n}$.

By aid of the substitution $u = \sqrt{2}\,\beta_n \sin\varphi$ we obtain:

$$I_{1,n} = \int_0^{\pi/2-\gamma_n/\beta_n} \frac{2\beta_n \cos\varphi \, d\varphi}{2\beta_n^2\cos^2\varphi - \gamma_n^2} =$$

$$= \sqrt{2} \int_0^{\sqrt{2}\beta_n \cos\gamma_n/\beta_n} \frac{du}{\alpha_n^2 + \beta_n^2 - u^2} =$$

$$= \frac{1}{\sqrt{2}\sqrt{\alpha_n^2+\beta_n^2}} \left\{ \ln\left[\sqrt{\alpha_n^2+\beta_n^2} + \sqrt{2}\,\beta_n \cos\frac{\gamma_n}{\beta_n}\right] - \ln\left|\sqrt{\alpha_n^2+\beta_n^2} - \sqrt{2}\,\beta_n \cos\frac{\gamma_n}{\beta_n}\right| \right\} =$$

$$(12) \quad = \frac{1}{\sqrt{2}\sqrt{\alpha_n^2+\beta_n^2}} \left\{ \ln\left[\sqrt{\alpha_n^2+\beta_n^2} + \sqrt{2}\,\beta_n \cos\frac{\gamma_n}{\beta_n}\right]^2 - \ln\left| \alpha_n^2+\beta_n^2 - 2\beta_n^2 \cos^2\frac{\gamma_n}{\beta_n}\right| \right\} .$$

It is easily seen that the first term of $\frac{1}{2^{n-1}} I_{1,n}$ tends to zero for $n \to \infty$. As to the second term, we remark that

$$\alpha_n^2 + \beta_n^2 - 2\beta_n^2\cos^2\frac{\gamma_n}{\beta_n} = \alpha_n^2 - \beta_n^2\cos 2\frac{\gamma_n}{\beta_n} = \gamma_n^2\left\{1 + 2\cos(2\vartheta\frac{\gamma_n}{\beta_n})\right\}$$

with $|\vartheta| \leq 1$.

Since $\lim_{n\to\infty} \ln\left\{1 + 2\cos(2\vartheta\frac{\gamma_n}{\beta_n})\right\} = 0$

we obtain finally

$$\lim_{n\to\infty} \frac{1}{2^{n-1}} I_{1,n} = -\lim_{n\to\infty} \frac{\sqrt{2}\ \ln\gamma_n}{2^{n-1}\sqrt{\alpha_n^2+\beta_n^2}} .$$

Hence we obtain by aid of formula (9)

$$\frac{\pi}{2M(a,b)} = -\lim_{n\to\infty} \frac{\sqrt{2}\ \ln \gamma_n}{2^{n-1}\sqrt{\alpha_n^2+\beta_n^2}} + \lim_{n\to\infty} \frac{1}{2^{n-1}} I_{2,n} +$$

$$+ \lim_{n\to\infty} \frac{1}{2^{n-1}} I_{3,n} .$$

It will be shown in the appendix that we have the relations

$$\lim_{n\to\infty} \frac{1}{2^{n-1}} I_{2,n} = \lim_{n\to\infty} \frac{1}{2^{n-1}} I_{3,n} = 0$$

and therefore we get:

$$(13) \quad \frac{\pi}{2M(a,b)} = \frac{-1}{M(\alpha,\beta)} \lim_{n\to\infty} \left\{ \frac{1}{2^{n-1}} \ln \gamma_n \right\}.$$

We observe now, that $\gamma_n = (\alpha_n^2 - \beta_n^2)^{1/2} = (\alpha_{n-1}-\beta_{n-1})/2 =$

$$= \frac{\gamma_{n-1}^2}{4\alpha_n}$$

and therefore $\gamma_n = \dfrac{\gamma_0^{2^n}}{4^{2^n-1}\alpha_n\, \alpha_{n-1}^2 \cdots \alpha_1^{2^{n-1}}}$

Thus we have the formula:

$$\frac{1}{2^{n-1}} \ln \gamma_n = \frac{1}{2} \ln \frac{\gamma_0}{4} - \frac{1}{2^{n-1}} \ln(\alpha_n \cdot \alpha_{n-1}^2 \cdots \alpha_1^{2^{n-1}}) +$$

$$+ \frac{1}{2^{n-1}} \ln 4$$

and so we obtain the following result which relates $M(a,b)$ with $M(\alpha,\beta)$.

$$(14) \quad \frac{\pi}{2} \frac{M(\alpha,\beta)}{M(a,b)} = 2\ln \frac{4}{\sqrt{\alpha^2-\beta^2}} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ln \gamma_1$$

From this result we may obtain again the formula (4) very easily.

Let $a=1$, $b=\varepsilon$ then $\alpha = 1+\varepsilon$, $\beta = 1-\varepsilon$ and $\alpha^2-\beta^2 = 4\varepsilon$
Since $|1-\alpha_n| < 2\varepsilon$ for all α_n, we have

$$\lim_{\varepsilon \to 0} M(\alpha,\beta) = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ln \alpha_n = 0$$

Substituting these results into (14) we get

$$(4) \qquad \lim_{\varepsilon \to 0} M(1,\varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2} .$$

Appendix

It will now be shown, that $\lim_{n\to\infty} \frac{1}{2^{n-1}} I_{2,n} =$

$$= \lim_{n\to\infty} \frac{1}{2^{n-1}} I_{3,n} = 0.$$

In order to do this we make use of the following lemma

Lemma Let

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8}(1+\theta x)^{-3/2}$$

then $\min_{0 \leq x \leq 1+\varepsilon} \theta > 0$ where ε is a sufficiently small positive number.

Proof: We remark that

$$(15) \quad \theta = x^{-1}\left\{\frac{x^{4/3}}{4}\left(1 + \frac{x}{2} - \sqrt{1+x}\right)^{-2/3} - 1\right\},$$

Thus θ is a continuous one-valued function of the real variable x when $x > 0$.

It can easily be shown that θ is positive for $0 < x \leq 1+\varepsilon$.

By proving that

$$\lim_{x\to 0} \theta > 0$$

it follows that θ assumes a positive minimum on the interval $0 \leq x \leq 1+\varepsilon$.

The Taylor series for $(1+x)^{1/2}$ is

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + 0\,(x^4),$$

We thus obtain, substituting this into (15),

$$\Theta = x^{-1}\left\{\left(\frac{x^{4/}}{4}\ \frac{x^2}{8} - \frac{x^3}{16} + O(x^4)\right)^{-2/3} - 1\right\} =$$

$$= x^{-1}\left\{\left(1 - \frac{x}{2} + O(x^2)\right)^{-2/3} - 1\right\} =$$

$$= x^{-1}\left\{\frac{x}{3} + O(x^2)\right\} = \frac{1}{3} + O(x)$$

which proves the lemma, moreover it follows that

$$(16) \qquad \min_{0 \le x \le 1+\varepsilon} \Theta \le \frac{1}{3} \ .$$

We proceed now to estimate $I_{2,n}$.

If, $0 \le \varphi \le \pi/2 - \beta_n/\gamma_n$ then

$$\frac{\gamma_n^2}{\beta_n^2 \cos^2\varphi} \le \frac{\gamma_n^2}{\beta_n^2}\left\{\frac{\gamma_n}{\beta_n} - \frac{\gamma_n^3}{6\beta_n^3} + \dots\right\}^{-2} = 1 + O(\gamma_n^2),$$

Thus for sufficiently large n we may state that

$$\frac{\gamma_n^2}{\beta_n^2 \cos^2\varphi} \le 1 + \varepsilon$$ where ε is a small positive number.

From the lemma, proved above, follows the existence of a positive minimum μ for Θ in formula (11), when φ varies from 0 to $\frac{\pi}{2} - \beta_n/\gamma_n$; moreover we know from (16) that $\mu < 1/2$.

Taking all this into consideration, we can estimate $I_{2,n}$ as follows

$$I_{2,n} = \int_0^{\pi/2 - \beta_n/\gamma_n} \frac{\gamma_n^4}{8\beta_n^3 \cos^3\varphi \left(1 + \Theta \frac{\gamma_n^2}{\beta_n^2 \cos^2\varphi}\right)^{3/2} \left(\beta_n \cos\varphi + \frac{\gamma_n^2}{2\beta_n \cos\varphi}\right)} \cdot \frac{1}{(\beta_n^2 \cos^2\varphi + \gamma_n^2)^{1/2}}\, d\varphi <$$

$$< \quad \frac{\gamma_n^4}{8} \int_0^{\pi/2} \frac{\beta_n \cos\varphi \, d\varphi}{(\beta_n^2 \cos^2\varphi + \mu \gamma_n^2)^3} =$$

$$= - \frac{\gamma_n^4}{8} \int_0^{\beta_n} \frac{du}{(u^2 - (\beta_n^2 + \mu \gamma_n^2))^3}$$

A trivial calculation *) shows that the last integral is equal to

$$\frac{\gamma_n^4}{8} \left[\frac{\beta_n}{4\delta_n^2(\delta_n^2-\beta_n^2)^2} + \frac{3\,\beta_n}{8\delta_n^4(\delta_n^2-\beta_n^2)} + \frac{3}{16\delta_n^5} \ln \frac{(\delta_n+\beta_n)^2}{\delta_n^2-\beta_n^2} \right],$$

where we have substituted $\delta_n^2 = \beta_n^2 + \mu\gamma_n^2$.

For sufficiently large n the following estimate holds:

$$\ln \frac{1}{\delta_n^2 - \beta_n^2} < \frac{1}{\delta_n^2 - \beta_n^2} < \frac{1}{(\delta_n^2 - \beta_n^2)^2} .$$

Therefore we may write

$I_{2,n} < K \dfrac{\gamma_n^4}{(\delta_n^2-\beta_n^2)^2} = \dfrac{K}{\mu^2}$ where K is some positive constant.

From this it follows immediately that:

$$\lim_{n\to\infty} \frac{1}{2^{n-1}} I_{2,n} = 0.$$

Finally we estimate $I_{3,n} = \int_{\pi/2-\gamma_n/\beta_n}^{\pi/2} \gamma_n (\beta_n^2 \cos^2\varphi + \gamma_n^2)^{-1/2} \, d\varphi.$

The length of the integration interval is γ_n/β_n, the maximum value of the integrand $1/\gamma_n$, and hence

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*) see e.g. H.B.Dwight: Tables of Integrals and other Mathematical Data. The MACMILLAN COMPANY. New York, 1955. p. 30.

$$I_{3,n} \leq \frac{\gamma_n}{\beta_n} \cdot \frac{1}{\gamma_n} = \frac{1}{\beta_n} ,$$

and thus

$$\lim_{n \to \infty} \frac{1}{2^{n-1}} I_{3,n} = 0 .$$